J Glob Optim (2010) 48:447–463
DOI 10.1007/s10898-009-9499-7

Semidefinite approximations for quadratic programs over orthogonal matrices

Janez Povh

Received: 4 October 2008 / Accepted: 17 November 2009 / Published online: 1 December 2009

Abstract Finding global optimum of a non-convex quadratic function is in general a very difficult task even when the feasible set is a polyhedron. We show that when the feasible set of a quadratic problem consists of orthogonal matrices from $\mathbb{R}^{n\times k}$, then we can transform it into a semidefinite program in matrices of order kn which has the same optimal value. This opens new possibilities to get good lower bounds for several problems from combinatorial optimization, like the *Graph partitioning problem* (GPP), the *Quadratic assignment problem* (QAP) etc. In particular we show how to improve significantly the well-known Donath-Hoffman eigenvalue lower bound for GPP by semidefinite programming. In the last part of the paper we show that the copositive strengthening of the semidefinite lower bounds for GPP and QAP yields the exact values.

Keywords Quadratic programming · Semidefinite programming · Copositive programming · Eigenvalue bound · Quadratic assignment problem · Graph partitioning problem

Mathematics Subject Classification (2000) 90C20 · 90C22 · 90C26 · 90C27

1 Introduction

Non-convex quadratic problems are a common research topic since they appear very often in combinatorial optimization. They are in general very hard, since already the following simple non-convex quadratic problem called the Standard quadratic programming program

$$\min\left\{x^T Qx : x \in \mathbb{R}^n_+, \sum_i x_i = 1\right\} \tag{1}$$

Supported by the Slovenian Research Agency (project no. 1000-08-210518).

J. Povh (✉)
Institute of Mathematics, Physics and Mechanics Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia
e-mail: janez.povh@fis.unm.si

is NP-hard to solve. More specifically, when $Q = A + I$ and A is the adjacency matrix of a graph G, then the optimal value of (1) yields the stability number of G (see [20]) which is NP-hard to compute.

In this paper we consider the following general non-convex quadratic program:

$$OPT_{QP} = \min \left\{ \mathrm{trace}(X^T AXB) : \; X \in \mathbb{R}_+^{n \times k}, \; X^T X = M, \; Q(X) = q \right\}, \qquad (QP)$$

where A and B are arbitrary symmetric matrices, M is a diagonal matrix and $Q(X) = q$ denotes some additional quadratic constraints. Several well-known NP-hard problems can be restated in the form QP, e.g. the Quadratic assignment problem, the Graph partitioning problem, the Weighted sums of eigenvalues problem etc.

The problems listed above are very tough, and there is no polynomial time algorithm which finds the optimal solution of these problems (unless P=NP). Optimal solutions are often computed with a branch and bound algorithm which has the exponential time complexity. The efficiency of this algorithm strongly depends on the quality of upper and lower bounds for the optimal value of the problem. Upper bounds we get with any heuristic, while computing lower bounds typically consists in relaxing some hard constraint and computing the optimal value of the relaxed problem.

Many researchers studied relaxations which yield spectral lower bounds. Hoffman and Wielandt [17] established an eigenvalue lower bound for the optimal value of QP for the case when the feasible set consists of non-negative square orthonormal matrices. They dropped the sign constraint and computed the optimal value of the relaxed problem which is determined by the eigenvalues of A and B, see Sect. 2. This is also known as the eigenvalue lower bound for the Quadratic assignment problem.

Donath and Hoffman [12] presented an eigenvalue lower bound for the Graph partitioning problem which is another special case of QP. They again relaxed the original problem which is NP-hard by ignoring the sign constraint and reformulated the resulting problem as an eigenvalue optimization problem.

Helmberg et al. [16] and Rendl and Wolkowicz [26] studied the projected eigenvalue lower bounds for the minimum cut problem and graph partitioning problems.

Anstreicher and Wolkowicz [3] reformulated the Hoffman-Wielandt and the Donath-Hoffman lower bounds as optimal values of semidefinite programs. A similar result was obtained by Povh and Rendl for the eigenvalue lower bound from [16], see [22,23]. These results are very important because further strengthenings of the eigenvalue lower bounds lead to untractable problems, while the semidefinite reformulation enables adding additional constraints and therefore improving the bounds.

Our contribution to the literature on approximation of non-convex quadratic programs consists of the following results:

- In Sect. 2 we prove a representation theorem which states that the Lagrangian relaxation of the quadratic program over the set of orthogonal matrices which is a semidefinite program is tight, if we add on the primal side a redundant semidefinite constraint. This results generalizes the Anstreicher–Wolkowicz result from [3].
- In Sect. 3 we present the implications of the representation theorem from Sect. 2 on computing lower bounds for the Graph partitioning problem—GPP (called sometimes also min-k-cut problem) and the Quadratic assignment problem (QAP). We show in particular that the Donath-Hoffman lower bound for GPP arises as the Lagrangian relaxation of the properly relaxed Graph partitioning problem, but there is in general non-zero duality gap in the relaxation. We propose some new constraints which improve Donath-Hoffman eigenvalue lower bound and demonstrate the power of this approach on some instances

of random graphs and compare the new semidefinite bounds with the strongest semidefinite bound from the literature [27]. Comparison suggests the conjecture that the new semidefinite bound OPT_{new2} is equal to the PRGP bound from [27].

- In Sect. 3.1 we analyze the contribution of our approach to the Quadratic assignment problem, which might be considered as a special case of GPP. We show that semidefinite bounds for GPP, when applied to QAP, coincide with some recently published bounds [24].
- We present that replacing the semidefinite constraint by completely positive constraint improves the lower bound and in the case of GPP and QAP yields even the exact value. This is the contents of Sect. 5.

1.1 Notation

We denote the ith standard unit vector by e_i. The vector of all ones is $u_n \in \mathbb{R}^n$ (or u, if the dimension n is obvious). The square matrix of all ones is J_n (or J), the identity matrix is I (or I_n) and $E_{ij} = e_i e_j^T$.

In this paper we consider the following sets of matrices:

- The vector space of real symmetric $n \times n$ matrices: $\mathcal{S}_n = \left\{X \in \mathbb{R}^{n\times n} : X = X^T\right\}$,
- the cone of $n \times n$ positive semidefinite matrices: $\mathcal{S}_n^+ = \left\{X \in \mathcal{S}_n : y^T X y \geq 0 \forall y \in \mathbb{R}^n\right\}$,
- the cone of $n \times n$ copositive matrices: $\mathcal{C}_n = \left\{X \in \mathcal{S}_n : y^T X y \geq 0 \forall y \in \mathbb{R}^n_+\right\}$,
- the cone of $n \times n$ completely positive matrices: $\mathcal{C}_n^* = \text{conv}\left\{yy^T : y \in \mathbb{R}^n_+\right\}$.

We also use $X \succeq 0$ for $X \in \mathcal{S}_n^+$. A linear program over $\mathbb{R}^n_+$ is called a linear program, a linear program over $\mathcal{S}_n^+$ is called a semidefinite program, while a linear program over $\mathcal{C}_n$ or $\mathcal{C}_n^*$ is called a copositive program.

The sign $\otimes$ stands for the Kronecker product. When we consider the matrix $X \in \mathbb{R}^{m\times n}$ as a vector from $\mathbb{R}^{mn}$, we write this vector as $\text{vec}(X)$ or x. By $\langle \cdot, \cdot \rangle$ we denote the standard scalar product, i.e. $\langle u, v \rangle = u^T v$ for $u, v \in \mathbb{R}^n$, and for $X, Y \in \mathbb{R}^{m\times n}$ we have $\langle X, Y \rangle = \text{trace}(X^T Y)$. For matrix columns and rows we use the Matlab notation: $X(i, :)$ and $X(:, i)$ stand for ith row and column, respectively. If $a \in \mathbb{R}^n$, then $\text{Diag}(a)$ is an $n \times n$ diagonal matrix with a on the main diagonal and $\text{diag}(X)$ is the main diagonal of a square matrix X.

For a matrix $Z \in \mathcal{S}_{kn}$ we often use the following block notation:

$$Z = \begin{bmatrix} Z^{11} & \cdots & Z^{1k} \\ \vdots & \ddots & \vdots \\ Z^{k1} & \cdots & Z^{kk} \end{bmatrix}, \tag{2}$$

where $Z^{ij} \in \mathbb{R}^{n\times n}$.

When P or $P_{\text{subscript}}$ is the name of the optimization problem, then OPT_P or $OPT_{\text{subscript}}$, respectively, denote their optimal values.

2 Semidefinite programming relaxations for QP

2.1 Representation theorem

Hoffman and Wielandt [17] showed that

$$OPT_{HW} = \min\left\{\langle X, AXB \rangle : X \in \mathbb{R}^{n\times n}, X^T X = I\right\} = \langle \lambda, \sigma \rangle_- \tag{3}$$

where λ and σ are the vectors of eigenvalues of A and B, respectively, and $\langle \lambda, \sigma \rangle_-$ denotes the scalar product, where we first sort the components of λ increasingly and the components of σ decreasingly. OPT_{HW} is a lower bound for OPT_{QP}, since the problem (3) is obtained from QP by omitting sign constraints and the quadratic constraint $Q(X) = q$.

Anstreicher and Wolkowicz [3] formulated this lower bound as the optimal value of a semidefinite program. They added in problem (3) the redundant constraint $XX^T = I$ and then considered the Lagrangian dual of the problem which is semidefinite program (4). They showed that strong duality holds for this case, hence we have

$$OPT_{HW} = \max\,\{\mathrm{trace}(S) + \mathrm{trace}(T)\colon S \in \mathcal{S},\ T \in \mathcal{S},\ S \otimes I + I \otimes T \preceq B \otimes A\}. \quad (4)$$

In this section we extend this result to a more general case when the matrices $X \in \mathbb{R}^{n\times k}$ still have orthonormal columns but are not square. In this case we have $X^T X = I_k$, but the other constraint $XX^T = I_n$ is not satisfied, if $k < n$. Therefore we can not repeat the Anstreicher-Wolkowicz procedure. The following lemma shows how the constraint $XX^T = I_n$ should be generalized to close the duality gap.

Lemma 1 *If $X \in \mathbb{R}^{n\times k}$ and $X^T X = I_k$, then $XX^T \preceq I_n$.*

Proof Every eigenvector for XX^T is either linear combination of columns of X or orthogonal to the columns of X, hence the eigenvalues of XX^T are either 0 or 1. □

Now we can prove the theorem.

Theorem 2 *Let $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_k$ and*

$$OPT_1 := \min\,\{\langle X,\ AXB\rangle\colon X \in \mathbb{R}^{n\times k},\ X^T X = I_k\} \quad (5)$$

$$OPT_2 := \max\,\left\{\mathrm{trace}(S) - \mathrm{trace}(T)\colon S \in \mathcal{S}_k,\ T \in \mathcal{S}_n^+,\ S \otimes I_n - I_k \otimes T \preceq B \otimes A\right\} \quad (6)$$

Then $OPT_1 = OPT_2$.

Proof Firstly we show that the problem (6) is the Lagrangian relaxation of (5), if we add the seemingly redundant constraint $XX^T \preceq I$. In the sequel we reduce (6) to a linear program and finally prove that the optimal value of the dual linear program is

$$\sum_i \sigma_i \lambda_{\varphi(i)}$$

where φ is some injection from $\{1,\ldots,k\}$ into $\{1,\ldots,n\}$ and $\lambda,\ \sigma$ are vectors with the eigenvalues of A and B, resp. This is at least OPT_1 [16, Theorem 5], completing the chain of inequalities.

We introduce the dual variable S for the constraint $X^T X = I_k$ and the dual variable T for the newly added constraint $XX^T \preceq I_n$. Clearly $S \in \mathcal{S}_k$, $T \in \mathcal{S}_n^+$ and we have

$$\begin{aligned} OPT_1 &= \min\{\langle X,\ AXB\rangle\colon X \in \mathbb{R}^{n\times k}, X^T X = I_k,\ XX^T \preceq I_n\} \\ &= \min_{X\in\mathbb{R}^{n\times k}} \left\{ \max_{S\in\mathcal{S}_k,\ T\in\mathcal{S}_n^+} \{\langle X,\ AXB\rangle + \langle S,\ I_k - X^T X\rangle - \langle T,\ I_n - XX^T\rangle\}\right\} \\ &\geq \max_{S\in\mathcal{S}_k,\ T\in\mathcal{S}_n^+} \left\{\mathrm{trace}(S) - \mathrm{trace}(T) + \min_{x\in\mathbb{R}^{nk}} x^T(B\otimes A - S\otimes I_n + I_k \otimes T)x\right\} \\ &= \max\,\left\{\mathrm{trace}(S) - \mathrm{trace}(T)\colon S \in \mathcal{S}_k,\ T \in \mathcal{S}_n^+,\ S\otimes I_n - I_k \otimes T \preceq B\otimes A\right\} \\ &= OPT_2\,. \end{aligned}$$

The first inequality follows from exchanging min and max and the last equality is due to the inner minimization problem which is a quadratic unconstrained problem and is therefore bounded from below if and only if its Hessian $B \otimes A - S \otimes I_n + I_k \otimes T$ is positive semidefinite. We also used the fact that

$$\langle X,\ AXB\rangle = x^T(B \otimes A)x \text{ for } x = \text{vec}(X).$$

We show that there is an equality above by transforming the last semidefinite program into a linear program. Since A and B are symmetric, we can find an orthonormal eigen decomposition $A = P\Lambda P^T$ and $B = Q\Sigma Q^T$, where $\Lambda = \text{Diag}(\lambda)$, $\Sigma = \text{Diag}(\sigma)$ and λ, σ are vectors with eigenvalues. We can write

$$\begin{aligned} OPT_2 &= \max \left\{ u_k^T s - u_n^T t \colon s \in \mathbb{R}^k,\ t \in \mathbb{R}^n_+,\ s_i - t_j \le \sigma_i \lambda_j,\ \forall i, j \right\} \\ &= \min \left\{ \sum_{i,j} \sigma_i \lambda_j z_{ij} \colon y \in \mathbb{R}^n_+,\ Z \in \mathbb{R}^{k\times n}_+,\ Zu_n = u_k,\ Z^T u_k + y = u_n \right\}. \end{aligned}$$

The first equality in the expression above follows from the fact that the cost function depends only on diagonal entries of the matrices S and T, so we may ignore all non-diagonal entries and write $s = \text{diag}(S)$ and $t = \text{diag}(T)$. The last optimization problem is a dual linear program to the last but one problem. We should note that the system matrix in the last linear program is totally unimodular, hence there exists (see [21]) an integer optimal solution

$$(Z^*, y^*) \in \mathbb{R}^{k\times n}_+ \times \mathbb{R}^n_+.$$

The matrix Z^* is therefore a 0-1 matrix and defines an injection $\varphi^* \colon \{1, \ldots, k\} \to \{1, \ldots, n\}$ with $\varphi^*(i) = j \iff z^*_{ij} = 1$. This means that we have proved

$$OPT_2 = \sum_{i=1}^{k} \sigma_i \lambda_{\varphi^*(i)} \ge \min \left\{ \sum_i \sigma_i \lambda_{\varphi(i)} \colon \varphi \text{ injection } \colon \{1, \ldots, k\} \to \{1, \ldots, n\} \right\}.$$

The optimal value of the right-hand side problem above is exactly OPT_1 (see e.g. [16, Theorem 5]), and from all relations from the beginning we can conclude that $OPT_1 = OPT_2$. □

Remark 1 If $k = n$, then (4) and (6) are equivalent. Indeed, if $(S^*,\ T^*) \in \mathcal{S}_k \times \mathcal{S}_n$ is feasible for (4), then $(\bar{S}, \bar{T})$ defined by $\bar{S} = S + \lambda_{\max}(T)$ and $\bar{T} = -T + \lambda_{\max}(T)$ is a feasible solution for (6) with the same objective value. Similarly any feasibly pair (S, T) for (6) gives a feasible solution $(S, -T)$ for (4) with the same objective value.

2.2 SDP relaxations for QP

Suppose that we have in QP only "pure" quadratic constraints, i.e. $Q(X) = q$ contains only equations of type $\langle X, A_i X B_i\rangle = q_i$, where A_i and B_i are arbitrary matrices such that the scalar product is defined. Based on the Theorem 2 we can obtain the following semidefinite lower bound for the optimal value of QP:

$$\begin{aligned} OPT_{QP} \ge \min\ & \text{trace}(X^T AXB) \\ & X \in \mathbb{R}^{n\times k}, \quad X^T X = M = \text{Diag}(m), \\ & \langle X, A_i X B_i\rangle = q_i, \quad 1 \le i \le p \\ = \min\ & \text{trace}\left(Y^T AYM^{1/2}BM^{1/2}\right) \end{aligned} \tag{7}$$

$$
\begin{aligned}
& Y \in \mathbb{R}^{n \times k}, \quad Y^T Y = I \\
& \langle Y, A_i Y M^{1/2} B_i M^{1/2} \rangle = q_i, \quad 1 \le i \le p \qquad (8) \\
\ge\ & \max\ \operatorname{trace}(S) - \operatorname{trace}(T) + q^T y \\
& S \in \mathcal{S}_k,\ T \in \mathcal{S}_n^+,\ y \in \mathbb{R}^p \\
& S \otimes I_n - I_k \otimes T + \sum_{i=1}^{p} y_i \left(M^{1/2} B_i^T M^{1/2}\right) \otimes A_i \preceq \left(M^{1/2} B M^{1/2}\right) \otimes A \qquad (9) \\
=\ & \min \langle M^{1/2} B M^{1/2} \otimes A,\ Z \rangle \\
& Z \in \mathcal{S}_{kn}^+,\ W \in \mathcal{S}_n^+ \\
& \sum_{i=1}^{k} Z^{ii} + W = I, \quad \langle I,\ Z^{ij} \rangle = \delta_{ij}, \quad 1 \le i, j \le k \\
& \langle M^{1/2} B_i^T M^{1/2} \otimes A_i, Z \rangle = q_i, \quad 1 \le i \le p \qquad (10) \\
=\ & \min\ \langle B \otimes A,\ V \rangle \\
(QP_{SDP}) \quad & V \in \mathcal{S}_{kn}^+,\ W \in \mathcal{S}_n^+ \\
& \sum_{i=1}^{k} \frac{1}{m_i} V^{ii} + W = I, \quad \langle I,\ V^{ij} \rangle = m_i \delta_{ij}, 1 \le i, j \le k \\
& \langle B_i^T \otimes A_i, V \rangle = q_i, \quad 1 \le i \le p
\end{aligned}
$$

Problem (7) is obtained from QP by omitting the sign constraint; (8) is obtained from (7) by substitution $Y = XM^{-1/2}$; (9) is the Lagrangian dual of (8); (10) is the Lagrangian dual of (9). The last semidefinite program (QP_{SDP}) is obtained from (10) by reverse substitution $V = (M^{1/2} \otimes I)Z(M^{1/2} \otimes I)$.

The first inequality appears since we dropped out the sign constraint in the quadratic problem QP, while the second inequality is due to the fact that the Lagrangian relaxation of non-convex quadratic program might have non-zero gap. This actually happens very often (see also Sect. 3 and Example 1). If we do not have quadratic equations $\langle X, A_i X B_i \rangle = q_i$, then the second inequality is an equality due to Theorem 2.

Note that the Lagrangian semidefinite duals have zero duality gap, since $S = 0$, $T = \alpha I$ and $y = 0$ are strictly feasible solution for the first semidefinite program above, if $\alpha > 0$ is sufficiently large (i.e. the matrices T and $(M^{1/2} B M^{1/2}) \otimes A + I_k \otimes T$ are positive definite when $T = \alpha I$ for $\alpha > 0$ sufficiently large).

Remark 2 If quadratic constraint $Q(X) = q$ contains also linear terms, i.e. is of the form $\langle X, A_i X B_i \rangle + \langle C_i, X \rangle = q_i, C_i \neq O$, then we can repeat the procedure by adding a row and column to the matrix variable in the semidefinite program. Constraints $\langle X, A_i X B_i \rangle + \langle C_i, X \rangle = q_i$ transform into $\langle \tilde{D}, \tilde{Z} \rangle = q_i$, where

$$
\tilde{D} = \begin{bmatrix} 0 & \frac{1}{2}\operatorname{vec}(C_i)^T \\ \frac{1}{2}\operatorname{vec}(C_i) & B_i^T \otimes A_i \end{bmatrix} \quad \text{and} \quad \tilde{Z} = \begin{bmatrix} 1 & z^T \\ z & Z \end{bmatrix}.
$$

3 New semidefinite approximations for the graph partitioning problem

In this section we demonstrate how to use the semidefinite representation results from the previous section to obtain new lower bounds for the Graph partitioning problem. We also analyze the contribution of this approach to the Quadratic assignment problem.

The Graph partitioning problem (GPP) is a classical problem from combinatorial optimization. Given a simple undirected graph $G = (V, E)$ with $|V| = n$, a number of partitions $k > 1$ and a vector $m = (m_1, m_2, \ldots, m_k) \in \mathbb{N}^k$ with $1 \le m_1 \le m_2 \le \cdots \le m_k$, $\sum_i m_i = n$, we are interested in a partition $(S_1, S_2, \ldots, S_k)$ of the vertex set V such that $|S_i| = m_i$ and the total number of cut edges (i.e. edges between different sets) is minimum. We can find in the literature GPP also under the name Min-k-cut problem [15]. A special instance of this problem is the Graph bisection problem. It is known to be NP-hard [14] and only $\mathcal{O}(\log n)$ approximation algorithms are currently known for the general instances. However, when the graph instance is dense, there exists a PTAS for the Graph bisection problem as well as for a different variant of Min-k-cut problem [4].

Graph partitioning problem has several applications: floor planning, analysis of bottlenecks in communication networks, partitioning the set of tasks among processors in order to minimize the communication between processors etc. A comprehensive survey with results in this area up to 1995 is contained in [2].

Due to proven complexity of the problem there exist several approaches to compute the optimal solution of GPP. Many of them are based on Branch and Bound technique where good lower bound for the optimal value are essential. Several authors [18,19,27] have shown that semidefinite programming based lower bounds are among the strongest but also very expensive to compute. In this section we show how to obtain new reasonably good semidefinite lower bound for OPT_{GPP} by tracing the procedure from Sect. 2.2.

We may represent any partition into k blocks with prescribed sizes by a (partition) matrix $X \in \{0, 1\}^{n \times k}$, where $x_{ij} = 1$ if and only if the ith vertex belongs to the jth set. With this notation the total number of cut edges is exactly $0.5\langle X, AXB\rangle$, where A is the adjacency matrix of the graph (i.e. $a_{ij} = 1$ if (ij) is an edge and $a_{ij} = 0$ otherwise) and $B = J_k - I_k$. If L is Laplacian matrix of a graph, then it holds $0.5\langle X, AXB\rangle = 0.5\langle X, LX\rangle$.

The set of all $n \times k$ partition matrices may be described as

$$\mathcal{P}_{n,k}(m) = \left\{X \in \mathbb{R}_+^{n\times k} \colon X^T X = M, \operatorname{Diag}(XX^T) = u_n, M = \operatorname{Diag}(m)\right\}. \tag{11}$$

We may use several equations to define the partition matrices, but the equations from (11) are the most convenient for our bounding procedure.

Graph partitioning problem may be therefore formulated as

$$OPT_{GPP} = \min \left\{\frac{1}{2}\langle X, LX\rangle : X \in \mathcal{P}_{n,k}(m)\right\}. \tag{GPP}$$

We can write constraint $\operatorname{Diag}(XX^T) = u_n$ as $\langle X, E_{ii}X\rangle = 1$, $1 \le i \le n$, hence GPP is a special instance for QP. Procedure from Sect. 2.2 yields the following semidefinite lower bound for OPT_{GPP}:

$$
\begin{aligned}
OPT_{GPP} \geq OPT_{DH} := \min\ & \frac{1}{2}\langle I \otimes L,\ V\rangle \\
& V \in \mathcal{S}_{kn}^{+},\ W \in \mathcal{S}_{n}^{+} \\
& \sum_i \frac{1}{m_i} V^{ii} + W = I,\ \langle I,\ V^{ij}\rangle = m_i \delta_{ij}, \\
& \langle I \otimes E_{ii}, V\rangle = 1, \quad 1 \leq i \leq n
\end{aligned}
\qquad (\text{GPP}_{SDP})
$$

We denote the optimal value of the semidefinite program from above by OPT_{DH}, since it is exactly the Donath-Hoffman eigenvalue lower bound [12] for OPT_{GPP}, as follows from the following theorem.

Theorem 3 *Optimal value of GPP_{SDP} is exactly the Donath-Hoffman eigenvalue lower bound for OPT_{GPP}.*

Proof Anstreicher and Wolkowicz [3] showed that the OPT_{DH}, defined as

$$
OPT_{DH} = \max\left\{\frac{1}{2}\sum_{i=1}^{k} m_{k-i+1}\lambda_i(L+D) \colon D = \text{Diag}(d), u^T d = 0\right\} \qquad (12)
$$

where $\lambda_1(L+D) \leq \lambda_2(L+D) \leq \cdots \quad \lambda_n(L+D)$ are the eigenvalues of $L+D$, can be represented as the optimal solution of the following semidefinite program

$$
\begin{aligned}
OPT_{DH} = \max\ & \text{trace}(S) + \text{trace}(T) \\
& \bar{M} \otimes (L + \text{Diag}(v)) - I \otimes S - T \otimes I \succeq 0 \\
& u_n^T v = 0,\ v \in \mathbb{R}^n,\ S,\ T \in \mathcal{S}_n,
\end{aligned}
$$

where

$$
\bar{M} = \frac{1}{2}\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}.
$$

This semidefinite program has matrix variable of order $n^2 \times n^2$ and is therefore much larger than GPP_{SDP}. It is obviously strictly feasible, hence its dual semidefinite program (which is feasible) has the same optimal solution. Therefore we have

$$
\begin{aligned}
OPT_{DH} = \min\ & \langle \bar{M} \otimes L,\ Y\rangle \\
& Y \ \in\ \mathcal{S}_{n^2}^{+},\ \sum_{i=1}^{n} Y^{ii} \ =\ I,\ \sum_{i=1}^{k} m_i\,\text{diag}(Y^{ii}) \ =\ u, \\
& \text{trace}(Y^{ij}) \ =\ \delta_{ij}, \quad 1 \leq i \leq j \leq n.
\end{aligned}
$$

One can see that OPT_{DH} is determined only by blocks Y^{ij} for $1 \leq i, j \leq k$, therefore we can write

$$
\begin{aligned}
OPT_{DH} = \min\ & \frac{1}{2}\langle M \otimes L,\ Y\rangle \\
& Y \ \in\ \mathcal{S}_{kn}^{+},\ \sum_{i=1}^{k} Y^{ii} \ \preceq\ I,\ \sum_{i=1}^{k} m_i\,\text{diag}(Y^{ii}) \ =\ u, \\
& \text{trace}(Y^{ij}) \ =\ \delta_{ij}, \quad 1 \leq i \leq j \leq k,
\end{aligned}
$$

After introducing $V = (M^{1/2} \otimes I)Y(M^{1/2} \otimes I)$ we obtain exactly the problem GPP_{SDP}. □

We can strengthen the lower bound OPT_{DH} by adding further constraints to GPP which are redundant for the partition matrices, but become important on the dual side. A relevant candidate is the "total sum" constraint

$$\langle X,\ J_n X J_k\rangle = n^2 \tag{13}$$

since it captures the property that the total sum of entries of any partition matrix of order $n \times k$ is n. It implies in GPP_{SDP} the constraint $\langle J_{kn},\ V\rangle = n^2$ which is no more redundant, hence we obtain a new lower bound for OPT_{GPP} denoted by OPT_{new1}:

$$\begin{aligned} OPT_{\text{new1}} := \min\ & \frac{1}{2}\langle I \otimes L,\ V\rangle \\ & V \in \mathcal{S}^+_{kn},\ W \in \mathcal{S}^+_n, \\ & \sum_i \frac{1}{m_i} V^{ii} + W = I,\ \ \langle I,\ V^{ij}\rangle = m_i \delta_{ij}, \\ & \langle I \otimes E_{ii},\ V\rangle = 1,\ 1 \le i \le n, \\ & \langle J_{kn},\ V\rangle = n^2. \end{aligned}$$

We can further improve OPT_{new1} by exploring the orthogonality of the columns of partition matrices. If $X \in \mathcal{P}_{n,k}(m)$, it holds

$$x_{ij}x_{i\ell} = 0 \quad \forall i, j, \ell,\ j \neq \ell, \tag{14}$$

hence we obtain

$$\begin{aligned} OPT_{\text{new2}} := \min\ & \frac{1}{2}\langle I \otimes L,\ V\rangle \\ & V \in \mathcal{S}^+_{kn},\ W \in \mathcal{S}^+_n \\ & \sum_i \frac{1}{m_i} V^{ii} + W = I,\ \ \langle I,\ V^{ij}\rangle = m_i \delta_{ij}, \\ & \langle I \otimes E_{ii},\ V\rangle = 1, \quad 1 \le i \le n, \\ & \langle J_{kn},\ V\rangle = n^2, \\ & \langle E_{j\ell} \otimes E_{ii},\ V\rangle = 0 \quad \forall i, j,\ \ell,\ j \neq \ell. \end{aligned}$$

The detailed computational study of the proposed semidefinite lower bounds is out of the scope of this paper and is planned as a part of the ongoing research. Nevertheless, to demonstrate the relative strength of the proposed lower bounds we report in Table 1 lower bounds OPT_{DH}, OPT_{new1}, OPT_{new2} and PRGP bound from [27] on nine random graphs with 50 nodes, where the probability that there exists an edge between two nodes varies from 0.1 to 0.9 (the probability is included in the name of the instance in the first column). For each instance we took $m = (5, 10, 15, 20)$. The second column of the table contains the number of graph nodes, the third column contains the number of graph edges and the last four columns contain the lower bounds OPT_{DH}, OPT_{new1}, OPT_{new2} and PRGP.

Table 1 demonstrates that adding few constraints to the semidefinite formulation of the eigenvalue lower bound does not make the semidefinite problem much harder, but the optimal value is improved significantly. Lower bound OPT_{new1} is obtained by improving the eigenvalue bound OPT_{DH} by only one constraint (13). On the other hand further tightening by (14) which contains $nk(k-1)/2$ equations yields much smaller improvement comparing to the additional effort needed to compute OPT_{new2}. Here $k = 4 \ll n = 50$ and therefore (14) implies $6n = 300$ additional equations. Comparison of the last two columns reveals

Table 1 Semidefinite lower bounds for graph partitioning problem, where $m = (5, 10, 15, 20)$

Name	n	|E|	OPT_{DH}	OPT_{new1}	OPT_{new2}	PRGP [27]
g50.01	50	111	17.922	22.762	23.570	23.549
g50.02	50	256	81.956	95.920	99.983	99.423
g50.03	50	342	124.718	148.701	152.231	151.225
g50.04	50	478	204.303	236.697	242.578	242.063
g50.05	50	611	287.204	332.791	338.494	338.529
g50.06	50	759	378.250	440.780	443.184	442.966
g50.07	50	897	470.157	544.238	550.335	549.934
g50.08	50	984	530.486	615.035	620.326	620.168
g50.09	50	1098	618.867	719.456	722.990	722.270

that OPT_{new2} is very close to PRGP on this instances. This suggests the conjecture that the semidefinite bound OPT_{new2} is equal to the bound PRGP from [27].

3.1 Quadratic assignment problem

The Quadratic assignment problem (QAP) can be stated in the following way. Let Π be the set of $n \times n$ permutation matrices (a matrix X is a permutation matrix, if it corresponds to some permutation ϕ, i.e. $x_{ij} \in \{0, 1\}$ and $x_{ij} = 1 \iff \phi(i) = j$). For given real symmetric $n \times n$ matrices A and B we want to find a permutation matrix $X \in \Pi$ which gives

$$OPT_{QAP} = \min\{\langle X, AXB\rangle : X \in \Pi\}. \tag{QAP}$$

The QAP can be considered as special instance of the Graph partitioning problem. Indeed, when $k = n$ we have $m = u_n$ and $\mathcal{P}_{n,k}(m)$ is exactly the set of permutation matrices Π. Matrices A and B may be arbitrary in QAP, therefore we can not express the objective function in terms of Laplacian matrix as we did in GPP.

The QAP has been intensively studied through the period of last decades, see the QAP library [9] and the survey [1] for the references on the results about QAP. QAP is an NP-hard problem and tight lower bounds are very desirable for the same reason as in GPP.

In the last decade several people intensively studied the semidefinite lower bound for the QAP. Zhao et al. [28], Sotirov and Rendl [25] and Povh and Rendl [24] lifted the problem from the vector space $\mathbb{R}^{n\times n}$ to $\mathcal{S}^+_{n^2+1}$ or $\mathcal{S}^+_{n^2}$ and formulated several semidefinite programs which give increasingly tight lower bounds for the QAP. The computational results [8,25] show that these lower bounds are among the strongest but also the most expensive to compute (in practice they could solve these programs for $n \leq 35$). Recently De Klerk and Sotirov [11] used group symmetries to compute the strongest and the most expensive lower bounds for some special instances of QAP with n up to 128.

In this subsection we show that the semidefinite lower bounds for OPT_{QAP}, based on semidefinite lower bound for OPT_{GPP}, coincides with some lower bounds from the literature. Note that we have to keep the objective function $\langle X, AXB\rangle$.

The Donath-Hoffman lower bound for QAP has the following form:

$$OPT^{QAP}_{DH} = \min \frac{1}{2}\langle I \otimes L, V\rangle$$
$$V \in \mathcal{S}^+_{n^2}, W \in \mathcal{S}^+_n$$

$$\sum_i V^{ii} + W = I,\ \langle I,\ V^{ij}\rangle = \delta_{ij}, \qquad (QAP_{SDP})$$
$$\langle I \otimes E_{ii}, V\rangle = 1, \quad 1 \le i \le n$$

Note that if (V, W) are feasible for QAP_{SDP}, then it follows that

$$\text{trace}(W) = \text{trace}(I) - \sum_i \text{trace}(V^{ii}) = n - n = 0.$$

Matrix W is positive semidefinite with zero trace, hence $W = 0$ and we can eliminate it from QAP_{SDP}. Moreover, since $W = 0$ it follows that $\sum_i V^{ii} = I$ implies $\langle I \otimes E_{ii}, V\rangle = 1, 1 \le i \le n$.

Lower bound OPT_{DH} therefore reduces to

$$OPT_{DH}^{QAP} = \min \frac{1}{2}\langle I \otimes L,\ V\rangle$$
$$V \in \mathcal{S}^+_{n^2},$$
$$\sum_i V^{ii} = I, \quad \langle I,\ V^{ij}\rangle = \delta_{ij},$$

This is exactly the Hoffman–Wielandt lower bound for QAP, since this semidefinite program is the dual of (4). This is not surprising since our bounding procedure from Sect. 2.2 coincides with the Anstreicher-Wolkovicz approach when $k = n$. Further tightening of OPT_{DH}^{QAP} by the "total sum constraint" (13) and the "orthogonality constraint" (14) leads to semidefinite bounds which are also known from the literature. More precisely, adding (13) gives lower bound equal to the OPT_{AW+} from [24] and further inclusion of (14) yields lower bound, which is strictly between OPT_{AW+} and OPT_{ZKRW1} from [24].

4 Tightness of the proposed relaxations

In the previous sections we show how to obtain semidefinite lower bounds for OPT_{QP} and how to improve them by adding new constraints. In all cases we are interested in accurate and computationally cheap semidefinite relaxations.

There are two sources for the gap between the optimal values of QP and QP_{SDP}, as there are two inequalities in the chain of optimization problems (7)–(10) on page 451. If the sign constraint is redundant, then the first inequality is an equality, but this happens very rarely (actually we could not provide an example).

The second reason for the gap is the duality gap in the Lagrangian relaxation of the non-convex quadratic problem. In Theorem 2 we only proved that there is no duality gap, if there is no quadratic constraint in QP beside the orthogonality constraint $X^T X = M$.

As we could see in Sect. 3, additional (redundant) quadratic constraints become necessary in QP after dropping the sign constraint, since they decrease both sources of gap.

The following example demonstrates this situation.

Example 1 Let $G = K_3$ be the complete graph on 3 vertices and $m = (1, 2)$ be the partitioning vectors for the Graph partitioning problem. Obviously we have $OPT_{GPP} = 2$ and any partitioning matrix is optimal. As we proved in Sect. 3, the Donath-Hoffman lower bound

(12) is obtained as Lagrangian relaxation of the following quadratic problem

$$\min \left\{ \frac{1}{2}\langle X,\ LX\rangle :\ X \in \mathbb{R}^{n\times k},\ X^T X = M, \mathrm{Diag}(XX^T) = u_n \right\}$$

By a short exercise we can see that this quadratic problem has in our case the optimal value $3-\sqrt{2} \approx 1.586$, while $OPT_{DH} = 1.500$. Here we have both gaps non-zero. Further strengthenings of OPT_{DH} by adding (13) and (14) yield $OPT_{\text{new1}} = OPT_{\text{new2}} = OPT_{GPP} = 2$.

5 Approximating OPT_{QP} by copositive programming

Several authors have proved recently that some quadratic programs can be restated as linear programs over the cone of copositive or completely positive matrices (in the sequel we call such problems copositive programs). Bomze et al. [6] proved that the Standard quadratic programming problem can be rewritten as linear program over the cone of completely positive matrices. De Klerk and Pasechnik [10] reformulated the stability number problem as copositive program. Povh and Rendl [22, 23, 24] proved that the 3-partitioning problem and the Quadratic assignment problem have a copositive representation. The strongest representation result for the time being is due to Burer [7] who showed that any quadratic problem with linear constraints in binary and continuous variables can be rewritten as copositive program.

All problems mentioned above are NP-hard problems and rewriting them as copositive programs does not make them tractable, but only opens a new line of possible relaxations based on approximations of the copositive or completely positive cone.

In this section we show that the lower bounds for the quadratic program QP can also be improved by copositive programming.

As we pointed out in Sect. 4, there are two possible sources for the gap between the optimal values of QP and QP_{SDP}. The first is due to the fact that we eliminated the sign constraint $X \geq 0$ in QP before computing the Lagrangian relaxation of QP. If we keep the sign constraint, then we get in the Lagrangian relaxation the following copositive constraint:

$$\left(M^{1/2}BM^{1/2}\right) \otimes A - S \otimes I_n + I_k \otimes T - \sum_i y_i \left(M^{1/2}B_i^T M^{1/2}\right) \otimes A_i \in \mathcal{C}_{kn},$$

and in the last semidefinite program QP_{SDP} the constraint $V \in \mathcal{S}_{kn}^+$ rewrites into $V \in \mathcal{C}_{kn}^*$. The resulting copositive program is therefore

$$\begin{aligned} OPT_{CP} = \min\langle B \otimes A,\ V\rangle & \\ V \in \mathcal{C}_{kn}^*,\ W \in \mathcal{S}_n^+ & \qquad (QP_{CP}) \\ \sum_i \frac{1}{m_i} V^{ii} + W = I,\ \langle I,\ V^{ij}\rangle = m_i\delta_{ij}, & \quad 1 \leq i, j \leq k \\ \langle B_i^T \otimes A_i, V\rangle = q_i,\ 1 \leq i \leq p. & \end{aligned}$$

Obviously we have $OPT_{QP} \geq OPT_{CP} \geq OPT_{SDP}$. We wonder when we get the equality $OPT_{QP} = OPT_{CP}$. In the following example we present the case when there is strict inequality.

Example 2 Suppose $A = J_n$, $B = J_n - I_n$, $m = u_n$ and we have no constraint of the type $\langle X,\ A_i X B_i\rangle = q_i$. Therefore QP is a trivial Quadratic assignment problem, and for any feasible matrix X (which is a permutation matrix) we have $\langle X, AXB\rangle = n^2 - n$, hence

$OPT_{QP} = n^2 - n$. On the other hand the matrices $V = \frac{1}{n} I_{n^2}$ and $W = O_n$ are feasible for the copositive relaxation QP_{CP} with $\langle B \otimes A,\ V\rangle = 0$, hence we have the gap $OPT_{QP} - OPT_{CP} = n^2 - n$.

5.1 Copositive relaxation for graph partitioning problem is tight

When we consider the Graph partitioning problem (see Sect. 3 for the definition), then we can alternatively describe the feasible set of GPP, i.e. the set of all partition matrices by

$$\begin{aligned}\mathcal{P}_{n,k}(m) &= \left\{X \in \{0,1\}^{n\times k}\colon X^T X = M\right\}\\ &= \left\{X \in \mathbb{R}_+^{n\times k}\colon X^T X = M, \operatorname{Diag}(XX^T) = u_n, \langle X,\ J_n X J_k\rangle = n^2\right\}. \end{aligned} \tag{15}$$

Recall that in the second formulation we do not need the constraint $\langle X,\ J_n X J_k\rangle = n^2$, since it is implied by other constraints, but it becomes important in the copositive relaxation of the problem for the same reason as in the previous sections.

By repeating the procedure from the beginning of the section we obtain the following copositive lower bound for OPT_{GPP}

$$\begin{aligned} OPT_{GPP} \geq \min\ & \frac{1}{2}\langle I_k \otimes L,\ V\rangle\\ & V \in \mathcal{C}^*_{kn},\ W \in \mathcal{S}^+_n\\ & \sum_i \frac{1}{m_i} V^{ii} + W = I_n,\ \ \langle I_n,\ V^{ij}\rangle = m_i\delta_{ij} \quad \forall i, j\\ & \langle I_k \otimes E_{ii},\ V\rangle = 1 \quad \forall i, \langle J_{kn}, V\rangle = n^2. \end{aligned}$$

Note that the bound OPT_{CP} is actually a strengthening of the OPT_{new1} lower bound from Sect. 3, since we replace $V \in \mathcal{S}^+_{kn}$ by $V \in \mathcal{C}^*_{kn}$. A closer look reveals that this lower bound is tight, i.e. $OPT_{GPP} = OPT_{CP}$. We can say more: there is a strong relation between the set of all partition matrices $\mathcal{P}_{n,k}(m)$ and the feasible set for GPP_{CP}, as follows from the following lemma.

Lemma 4 *The following is equivalent*:

(a) $V \in \mathcal{C}^*_{kn}$ *is feasible for* GPP_{CP};
(b) $V = \sum_s \lambda_s p_s p_s^T$, *where* $\lambda_s \geq 0$, $\sum_s \lambda_s = 1$, $p_s = \operatorname{vec}(P_s)$ *and* $P_s \in \mathcal{P}_{n,k}(m), \forall s$.

Proof The implication (b) ⇒ (a) is easy: if P is a partition matrix, then pp^T is completely positive and is feasible for GPP_{CP}. The same is true for any convex combination of such matrices.

The direction (a) ⇒ (b) is more involved. Let $V \in \mathcal{C}^*_{kn}$ be feasible for GPP_{CP}. By definition we have $V = \sum_s q_s q_s^T$, where $q_s \in \mathbb{R}^{kn}_+$ and $q_s \neq 0\ \forall s$. We can obtain from each vector q_s a non-negative matrix Q_s such that $q_s = \operatorname{vec}(Q_s)$. The constraint $\langle I_n,\ V^{ij}\rangle = m_i\delta_{ij}$ implies that $\sum_s \langle I_n,\ Q_s(:, i) Q_s(:, j)^T\rangle = m_i\delta_{ij}$ or equivalently $\sum_s Q_s^T Q_s = M$. In particular this means that each Q_s has orthogonal columns, because all Q_s are non-negative.

The $n \times n$ matrix

$$\hat{V} = \sum_{i,j} V^{ij} = (u_k \otimes I_n)^T V (u_k \otimes I_n)$$

is positive semidefinite and satisfies $\operatorname{diag}(\hat{V}) = u_n$. Since we have $\langle J_n,\ \hat{V}\rangle = n^2$ it follows that $\hat{V} = J_n$ or equivalently $\langle J_k \otimes E_{ij}, V\rangle = 1$ for all i, j. Let us denote by $r_i(Q)$ the sum of i-th row of Q. Therefore we have for each pair $1 \leq i < j \leq n$:

$$\sum_s r_i(Q_s)^2 = \langle J_k \otimes E_{ii}, V \rangle = 1,$$
$$\sum_s r_j(Q_s)^2 = \langle J_k \otimes E_{jj}, V \rangle = 1,$$
$$\sum_s r_i(Q_s) r_j(Q_s) = \langle J_k \otimes E_{ij}, V \rangle = 1.$$

The Cauchy–Schwarz inequality for the equality case implies $r_i(Q_s) = \alpha r_j(Q_s)$ and $\alpha = 1$. Since we have $r_i(Q_s) = r_j(Q_s)$ for all $1 \leq i \leq j \leq n$, we can define $P_s = \frac{1}{r_1(Q_s)} Q_s$ and $\lambda_s = r_1^2(Q_s)$ (note that $r_i(Q_s) \neq 0\ \forall i, s$, since $q_s \neq 0\ \forall s$). It follows $V = \sum_s \lambda_s p_s p_s^T$ and $\sum_s \lambda_s = 1$ (with $p_s = \mathrm{vec}(P_s)$). We already have the factorization of V, stated by the lemma. To finish the proof, we need to show that P_s are partition matrices.

We know by definition that each $n \times n$ matrix P_s has non-negative entries, orthogonal columns and $r_i(P_s) = 1\ \forall i$. This implies $P_s \in \{0, 1\}^{n \times k}$ and $\sum_{i,j} P_s(i, j) = n$, as well as $\sum_i P_s(i, j) = \sum_i P_s(i, j)^2\ \forall j$. It remains to show that $\sum_i P_s(i, j) = m_j \forall j$. This will be done in the last part of the proof.

The constraint $\langle I_n, V^{jj} \rangle = \sum_s \lambda_s \sum_i P_s(i, j)^2 = m_j$ implies:

$$\langle J_n, V^{jj} \rangle = \sum_s \lambda_s \left(\sum_i P_s(i, j) \right)^2 \geq \left(\sum_s \lambda_s \sum_i P_s(i, j) \right)^2$$
$$= \left(\sum_s \lambda_s \sum_i P_s(i, j)^2 \right)^2 = m_j^2.$$

On the other hand, from $\sum_i \frac{1}{m_i} V^{ii} \preceq I_n$ it follows

$$\langle J_n, I_n \rangle = n \geq \sum_i \frac{1}{m_i} \langle J_n, V^{ii} \rangle \geq \sum_i \frac{1}{m_i} m_i^2 = n,$$

hence $\langle J_n, V^{jj} \rangle = m_j^2\ \forall j$.

The $k \times k$ matrix $\tilde{V} = \sum_{i,j} \langle J_n, V^{ij} \rangle E_{ij} = (I_k \otimes u_n)^T V (I_k \otimes u_n)$ is positive semidefinite with $\tilde{V}_{ii} = m_i^2$ and $\langle J_k, \tilde{V} \rangle = n^2$, hence it must hold $\tilde{V}_{i,j} = m_i m_j\ \forall i, j$, or equivalently $\langle J_n, V^{ij} \rangle = m_i m_j$. Using this property we obtain for each pair $1 \leq i < j \leq k$:

$$\langle J_n, V^{ii} \rangle = \sum_s \lambda_s \left(\sum_t P_s(t, i) \right)^2 = m_i^2,$$
$$\langle J_n, V^{jj} \rangle = \sum_s \lambda_s \left(\sum_t P_s(t, j) \right)^2 = m_j^2,$$
$$\langle J_n, V^{ij} \rangle = \sum_s \lambda_s \left(\sum_t P_s(t, i) \right) \left(\sum_t P_s(t, j) \right) = m_i m_j.$$

The Cauchy–Schwarz inequality again implies $\sum_t P_s(t, i) = \frac{m_i}{m_j} \sum_t P_s(t, j) \forall s$, hence, since the sum of all entries in each P_s is n, we obtain $\sum_t P_s(t, i) = m_i$, $1 \leq i \leq k$. $\square$

Note that we can write GPP also in the form

$$OPT_{GPP} = \min \left\{ \frac{1}{2} \langle I_k \otimes L,\ V \rangle : V = pp^T,\ P \in \mathcal{P}_{n,k}(m) \right\}$$
$$= \min \left\{ \frac{1}{2} \langle I_k \otimes L,\ V \rangle : V = \sum_s \lambda_s p_s p_s^T,\ \lambda_s \geq 0, \right.$$
$$\left. \sum_s \lambda_s = 1,\ p_s = \text{vec}(P_s),\ P_s \in \mathcal{P}_{n,k}(m) \right\}.$$

Lemma 4 implies that any feasible solution for GP_{CP} is feasible for the latter formulation of GPP and vice versa, hence the following theorem follows immediately.

Theorem 5 *For the Graph partitioning problem we have* $OPT_{GPP} = OPT_{CP}$.

Result from Theorem 5 a generalization of the result for the specific 3-partitioning problem which was obtained by Povh and Rendl [23]. The copositive formulation for 3-partitioning problem contains different set of equations as it was obtained by different techniques and different initial formulations.

Theorem 5 can be applied also to QAP yielding a copositive representation for QAP, which is essentially the same as the copositive formulation of QAP from [24].

6 Conclusions

Computing good lower bounds for hard problems from combinatorial optimization is very important, especially if we plan to solve the problem by Branch and Bound method. In the paper we present how to improve by semidefinite programming the eigenvalue lower bounds for some problems, where the feasible set consists of orthogonal matrices.

We rewrite the quadratic problem over the set of (non-quadratic) orthogonal matrices as a semidefinite program. This result is a generalization of the result from Anstreicher and Wolkowicz [3], and opens new possibilities for approximating some hard quadratic problems, where the feasible set consists of orthogonal matrices subject to some additional constraints (like the Quadratic assignment problem (QAP), the Graph partitioning problem (GPP), the Weighted sums of eigenvalues problem etc.). We offer a generic semidefinite program which gives tighter lower bounds for these problems, comparing to the eigenvalue lower bounds. The semidefinite bound can be further improved towards the optimum of the original problem by adding linear constraints which correspond to some valid constraint in the original problem. We also suggested few such constraints for QAP and GPP and they indeed improved the bounds significantly. We give some preliminary computational results which show the potential of this approach.

In the last section we demonstrate the power of copositive programming. If we replace the semidefinite constraint in the generic semidefinite program by completely positive constraint, we obtain better lower bounds. In the case of Graph partitioning problem the resulting copositive program even delivers the exact value.

We also try to address the question whether we can extend Theorem 2 to a larger class of quadratic problems at least to quadratic programs over the orthogonal matrices which satisfy some additional quadratic or linear constraints? While Beck [5] provided a weak but positive answer to the first question we show that the Graph partitioning problem certainly does not belong to any of these classes (see Example 1).

Acknowledgments We gratefully acknowledge the financial support by Slovene research agency under contract 1000-08-210518.

References

1. Anstreicher, K.: Recent advances in the solution of quadratic assignment problems. Math. Program. B **97**, 27–42 (2003)
2. Alpert, C.J., Kahng, A.B.: Recent directions in netlist partition: a survey. Integr. VLSI J. **19**, 1–81 (1995)
3. Anstreicher, K., Wolkowicz, H.: On lagrangian relaxation of quadratic matrix constraints. SIAM J. Matrix Anal. Appl. **22**, 41–55 (2000)
4. Arora, S., Karger, D., Karpinski, M.: Polynomial time approximation schemes for dense instances of NP-hard problems. J. Comput. Syst. Sci. **58**(1), 193–210 (1999)
5. Beck, A.: Quadratic matrix programming. SIAM J. Optim. **17**, 1224–1238 (2007)
6. Bomze, I., Duer, M., de Klerk, E., Roos, C., Quist, A.J., Terlaky, T.: On copositive programming and standard quadratic optimization problems. J. Glob. Optim. **18**, 301–320 (2000)
7. Burer, S.: On the copositive representation of binary and continuous nonconvex quadratic programs. Math. Program., published online on April 29th (2008)
8. Burer, S., Vandenbussche, D.: Solving lift-and-project relaxations of binary integer programs. SIAM J. Optim. **16**, 726–750 (2006)
9. Burkard, R.E., Cela, E., Karisch, S.E., Rendl, F.: QAPLIB—a quadratic assignment problem library. Available at http://www.seas.upenn.edu/qaplib/, July (2007)
10. de Klerk, E., Pasechnik, D.V.: Approximation of the stability number of a graph via copositive programming. SIAM J. Optim. **12**, 875–892 (2002)
11. de Klerk, E., Sotirov, R.: Exploiting group symmetry in semidefinite programming relaxations of the quadratic assignment problem. To appear in Math. Program., available on http://www.springerlink.com/content/85302n245v250051/fulltext.pdf.
12. Donath, W.E., Hoffman, A.J.: Lower bounds for the partitioning of graphs. IBM J. Res. Dev. **17**, 420–425 (1973)
13. Falkner, J., Rendl, F., Wolkowicz, H.: A computational study of graph partitioning. Math. Program. **66**(2), 211–240 (1994)
14. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-completeness. Freeman, San Francisco (1979)
15. Guttmann-Beck, N., Hassin, R.: Approximation algorithms for minimum K-cut. Algorithmica **27**, 198–207 (2000)
16. Helmberg, C., Rendl, F., Mohar, B., Poljak, S.: A spectral approach to bandwidth and separator problems in graphs. Linear Multilinear Algebra **39**, 73–90 (1995)
17. Hoffman, A.J., Wielandt, H.W.: The variation of the spectrum of a normal matrix. Duke Math. J. **20**, 37–39 (1953)
18. Karisch, S.E., Rendl, F.: Semidefinite programming and graph equipartition. In: Topics in Semidefinite and Interior-point Methods, The Fields Institute for Research in Mathematical Sciences, vol. 18. Communications Series, American Mathematical Society, Providence, RI (1998)
19. Lisser, A., Rendl, F.: Graph partitioning using linear and semidefinite programming. Math. Program. **95**(1), 91–101 (2003)
20. Motzkin, T.S., Straus, E.G.: Maxima for graphs and a new proof of a theorem of Túran. Can. J. Math. **17**, 533–540 (1965)
21. Papadimitriou, C.H., Steiglitz, K.: Combinatorial Optimization: Algorithms and Complexity. Dover Publications, Mineola, NY (1998)
22. Povh, J.: Application of semidefinite and copositive programming in combinatorial optimization. PhD Thesis, University of Ljubljana, Ljubljana (2006)
23. Povh, J., Rendl, F.: A copositive programming approach to graph partitioning. SIAM J. Optim. **18**, 223–241 (2007)
24. Povh, J., Rendl, F.: Copositive and semidefinite relaxations of the quadratic assignment problem. Discrete Optim. **6**, 231–241 (2009)
25. Rendl, F., Sotirov, R.: Bounds for the quadratic assignment problem using the bundle method. Math. Program. **109**(2–3), 505–524 (2007)
26. Rendl, F., Wolkowicz, H.: A projection technique for partitioning the nodes of a graph. Ann. Oper. Res. **58**, 155–179 (1995)

27. Wolkowicz, H., Zhao, Q.: Semidefinite programming relaxations for the graph partitioning problem. Discrete Appl. Math. **96–97**, 461–479 (1999)
28. Zhao, Q., Karisch, S.E., Rendl, F., Wolkowicz, H.: Semidefinite programming relaxations for the quadratic assignment problem. J. Comb. Optim. **2**, 71–109 (1998)